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Using a functional method it is demonstrated that a generic quantum system evolves to a decohered state in a final pointer basis.

I. INTRODUCTION.

We will demonstrate that for a wide set of quantum systems the quantum regime can be consider as the transient phase while the final classical equilibrium regime is the permanent state. We will find a basis where exact matrix decoherence appears for these final states. Therefore we will find a set of final intrinsically consistent histories.

II. DECOHERENCE.

A. Decoherence in the energy.

Let us consider a closed and isolated quantum system with $N + 1$ dynamical variables and a Hamiltonian endowed with a continuous spectrum and just one bounded ground state. So the discrete part of the spectrum of H has only one value ω_0 and the continuous spectrum is let say $0 \leq \omega < \infty$. Eventually we will give the collective name x to both ω_0 and ω . Let us assume that it is possible to diagonalize the Hamiltonian H , together with N observables O_i ($i = 1, \dots, N$). The operators (H, O_1, \dots, O_N) form a *complete set of commuting observables* (CSCO). For simplicity we also assume a discrete spectrum for the N observables O_i . Therefore we write

$$H = \omega_0 \sum_m |\omega_0, m\rangle \langle \omega_0, m| + \int_0^\infty \omega \sum_m |\omega, m\rangle \langle \omega, m| d\omega \quad (1)$$

where $\omega_0 < 0$ is the energy of the ground state, and $m \doteq \{m_1, \dots, m_N\}$ labels a set of discrete indexes which are the eigenvalues of the observables O_1, \dots, O_N . $\{|\omega_0, m\rangle, |\omega, m\rangle\}$ is a basis of simultaneous generalized eigenvectors of the CSCO:

$$H|\omega_0, m\rangle = \omega_0|\omega_0, m\rangle, \quad H|\omega, m\rangle = \omega|\omega, m\rangle, \\ O_i|\omega_0, m\rangle = m_i|\omega_0, m\rangle, \quad O_i|\omega, m\rangle = m_i|\omega, m\rangle.$$

The most general observable that we are going to consider in our model reads:

$$O = \sum_{mm'} O(\omega_0)_{mm'} |\omega_0, m\rangle \langle \omega_0, m'| + \sum_{mm'} \int_0^\infty d\omega O(\omega)_{mm'} |\omega, m\rangle \langle \omega, m'| + \\ + \sum_{mm'} \int_0^\infty d\omega O(\omega, \omega_0)_{mm'} |\omega, m\rangle \langle \omega_0, m'| + \\ + \sum_{mm'} \int_0^\infty d\omega' O(\omega_0, \omega')_{mm'} |\omega_0, m\rangle \langle \omega', m'| + \\ + \sum_{mm'} \int_0^\infty \int_0^\infty d\omega d\omega' O(\omega, \omega')_{mm'} |\omega, m\rangle \langle \omega', m'|, \quad (2)$$

where $O(\omega)_{mm'}$, $O(\omega, \omega_0)_{mm'}$, $O(\omega_0, \omega')_{mm'}$ and $O(\omega, \omega')_{mm'}$ are ordinary functions of the real variables ω and ω' (these functions must have some mathematical properties in order to develop the theory; these properties are listed in paper [1]). We will say that these observables belong to a space \mathcal{O} . This space has the *basis* $\{|\omega_0, mm'\rangle, |\omega, mm'\rangle, |\omega\omega_0, mm'\rangle, |\omega_0\omega', mm'\rangle, |\omega\omega', mm'\rangle\}$:

$$|\omega_0, mm'\rangle \doteq |\omega_0, m\rangle \langle \omega_0, m'|, \quad |\omega, mm'\rangle \doteq |\omega, m\rangle \langle \omega, m'|,$$

$$|\omega\omega_0, mm'\rangle \doteq |\omega, m\rangle\langle\omega_0, m'|, \quad |\omega_0\omega', mm'\rangle \doteq |\omega_0, m\rangle\langle\omega', m'|, \quad (3)$$

$$|\omega\omega', mm'\rangle \doteq |\omega, m\rangle\langle\omega', m'|$$

The quantum states ρ are measured by the observables just defined, computing the mean values of these observable in the quantum states, i. e. in the usual notation: $\langle O \rangle_\rho = \text{Tr}(\rho^\dagger O)$ [2]. These mean values, generalized as in paper [1], can be considered as linear functionals ρ (mapping the vectors O on the real numbers), that we can call $(\rho|O)$ [3]. In fact, this is a generalization of the usual mean value definition. Then $\rho \in \mathcal{S} \subset \mathcal{O}'$, where \mathcal{S} is a convenient convex set contained in \mathcal{O}' , the space of linear functionals over \mathcal{O} [4], [5]. The basis of \mathcal{O}' (that can also be considered as the *co-basis* of \mathcal{O}) is $\{(\omega_0, mm'|, (\omega, mm'|, (\omega\omega_0, mm'|, (\omega_0\omega', mm'|, (\omega\omega', mm'|)\}$ defined as functionals by the equations:

$$\begin{aligned} (\omega_0, mm'|\omega_0, nn') &= \delta_{mn}\delta_{m'n'}, & (\omega, mm'|\eta, nn') &= \delta(\omega - \eta)\delta_{mn}\delta_{m'n'}, \\ (\omega\omega_0, mm'|\eta\omega_0, nn') &= \delta(\omega - \eta)\delta_{mn}\delta_{m'n'}, \\ (\omega_0\omega', mm'|\omega_0\eta', nn') &= \delta(\omega' - \eta')\delta_{mn}\delta_{m'n'}, \\ (\omega\omega', mm'|\eta\eta', nn') &= \delta(\omega - \eta)\delta(\omega' - \eta')\delta_{mn}\delta_{m'n'}. \end{aligned} \quad (4)$$

and all other $(\cdot|\cdot)$ are zero. Then, a generic quantum state reads:

$$\begin{aligned} \rho &= \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} (\omega_0, mm'| + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} (\omega, mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega_0)}_{mm'} (\omega\omega_0, mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega' \overline{\rho(\omega_0, \omega')}_{mm'} (\omega_0\omega', mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{mm'} (\omega\omega', mm'| \end{aligned} \quad (5)$$

where

$$\overline{\rho(\omega, \omega_0)}_{mm'} = \rho(\omega_0, \omega)_{m'm}, \quad \overline{\rho(\omega, \omega')}_{mm'} = \rho(\omega', \omega)_{m'm},$$

and $\overline{\rho(\omega_0)}_{mm}$ and $\overline{\rho(\omega)}_{mm}$ are real and non negative satisfying the total probability condition

$$(\rho|I) = \sum_m \rho(\omega_0)_{mm} + \sum_m \int_0^\infty d\omega \rho(\omega)_{mm} = 1, \quad (6)$$

where $I = \sum_m |\omega_0, m\rangle\langle\omega_0, m| + \int_0^\infty d\omega \sum_m |\omega, m\rangle\langle\omega, m|$ is the identity operator in \mathcal{O} . Eq. (6) is the extension to state functionals of the usual condition $\text{Tr}\rho^\dagger = 1$, used when ρ is a density operator.

The time evolution of the quantum state ρ reads:

$$\begin{aligned} \rho(t) &= \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} (\omega_0, mm'| + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} (\omega, mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega_0)}_{mm'} e^{i(\omega - \omega_0)t} (\omega\omega_0, mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega' \overline{\rho(\omega_0, \omega')}_{mm'} e^{i(\omega_0 - \omega')t} (\omega_0\omega', mm'| + \\ &+ \sum_{mm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{mm'} e^{i(\omega - \omega')t} (\omega\omega', mm'| \end{aligned} \quad (7)$$

As we only measure mean values of observables in quantum states, i. e.:

$$\begin{aligned}
\langle O \rangle_{\rho(t)} &= (\rho(t)|O) = \\
&= \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} O(\omega_0)_{mm'} + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} O(\omega)_{mm'} + \\
&+ \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega, \omega_0)}_{mm'} e^{i(\omega - \omega_0)t} O(\omega, \omega_0)_{mm'} + \\
&+ \sum_{mm'} \int_0^\infty d\omega' \overline{\rho(\omega_0, \omega')}_{mm'} e^{i(\omega_0 - \omega')t} O(\omega_0, \omega')_{mm'} + \\
&+ \sum_{mm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{mm'} e^{i(\omega - \omega')t} O(\omega, \omega')_{mm'}, \tag{8}
\end{aligned}$$

using the Riemann-Lebesgue theorem we obtain the limit, for all $O \in \mathcal{O}$

$$\lim_{t \rightarrow \infty} \langle O \rangle_{\rho(t)} = \langle O \rangle_{\rho_*} \tag{9}$$

where we have introduced the diagonal asymptotic or equilibrium state functional

$$\rho_* = \sum_{mm'} \overline{\rho(\omega_0)}_{mm'} (\omega_0, mm'| + \sum_{mm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{mm'} (\omega, mm'| \tag{10}$$

Therefore, in a weak sense we have:

$$W \lim_{t \rightarrow \infty} \rho(t) = \rho_* \tag{11}$$

Thus, any quantum state goes weakly to a linear combination of the energy diagonal states $(\omega_0, mm'|$ and $(\omega, mm'|$ (the energy off-diagonal states $(\omega\omega_0, mm'|$, $(\omega_0\omega', mm'|$ and $(\omega\omega', mm'|$ are not present in ρ_*). This is the case if we observe and measure the system evolution with *any possible observable of space \mathcal{O}* . Then, from the observational point of view, we have decoherence of the energy levels, even that, from the strong limit point of view the off-diagonal terms never vanish, they just oscillate, since we cannot directly use the Riemann-Lebesgue theorem in the operator equation (7).

B. Decoherence in the other "momentum" dynamical variables.

Having established the decoherence in the energy levels we must consider the decoherence in the other dynamical variables O_i , of the CSCO where we are working. We will call these variables "momentum variables". For the sake of simplicity we will consider, as in the previous section, that the spectra of these dynamical variables are discrete. As the expression of ρ_* given in eq. (10) involve only the time independent components of $\rho(t)$, it is impossible that a different decoherence process take place to eliminate the off-diagonal terms in the remaining N dynamical variables. Therefore, the only thing to do is to find if there is a basis where the off-diagonal components of $\rho(\omega_0)_{mm'}$ and $\rho(\omega)_{mm'}$ vanish at any time before the equilibrium is reached.

Let us consider the following change of basis

$$|\omega_0, r\rangle = \sum_m U(\omega_0)_{mr} |\omega_0, m\rangle, \quad |\omega, r\rangle = \sum_m U(\omega)_{mr} |\omega, m\rangle,$$

where r and m are short notations for $r \doteq \{r_1, \dots, r_N\}$ and $m \doteq \{m_1, \dots, m_N\}$, and $[U(x)^{-1}]_{mr} = \overline{U(x)}_{rm}$ (x denotes either $\omega_0 < 0$ or $\omega \in \mathbb{R}^+$).

The new basis $\{|\omega_0, r\rangle, |\omega, r\rangle\}$ verify the generalized orthogonality conditions

$$\begin{aligned}
\langle \omega_0, r | \omega_0, r' \rangle &= \delta_{rr'}, \quad \langle \omega, r | \omega', r' \rangle = \delta(\omega - \omega') \delta_{rr'}, \\
\langle \omega_0, r | \omega, r' \rangle &= \langle \omega, r | \omega_0, r' \rangle = 0.
\end{aligned}$$

As $\overline{\rho(\omega_0)}_{mm'} = \rho(\omega_0)_{m'm}$ and $\overline{\rho(\omega)}_{mm'} = \rho(\omega)_{m'm}$, it is possible to choose $U(\omega_0)$ and $U(\omega)$ in such a way that the off-diagonal parts of $\rho(\omega_0)_{rr'}$ and $\rho(\omega)_{rr'}$ vanish, i.e.

$$\rho(\omega_0)_{rr'} = \rho_r(\omega_0) \delta_{rr'}, \quad \rho(\omega)_{rr'} = \rho_r(\omega) \delta_{rr'}.$$

Therefore, there is a *final pointer basis* for the observables given by $\{|\omega_0, rr'\rangle, |\omega, rr'\rangle, |\omega\omega_0, rr'\rangle, |\omega_0\omega', rr'\rangle, |\omega\omega', rr'\rangle\}$ and defined as in eq. (3). The corresponding final pointer basis for the states $\{(\omega_0, rr'|, (\omega, rr'|, (\omega\omega_0, rr'|, (\omega_0\omega', rr'|, (\omega\omega', rr'|)\}$ diagonalizes the time independent part of $\rho(t)$ and therefore it diagonalizes the final state ρ_*

$$\rho_* = W \lim_{t \rightarrow \infty} \rho(t) = \sum_r \rho_r(\omega_0) (\omega_0, rr'| + \sum_r \int_0^\infty d\omega \rho_r(\omega) (\omega, rr'|. \quad (12)$$

Now we can define the final *exact pointer observables* [9]

$$P_i = \sum_r P_r^i(\omega_0) |\omega_0, r\rangle \langle \omega_0, r| + \int_0^\infty d\omega \sum_r P_r^i(\omega) |\omega, r\rangle \langle \omega, r|. \quad (13)$$

As H and P_i are diagonal in the basis $\{|\omega_0, r\rangle, |\omega, r\rangle\}$, the set $\{H, P_i, \dots, P_N\}$ is precisely the complete set of commuting observables (CSCO) related to this basis, where ρ_* is diagonal in the corresponding co-basis for states. For simplicity we define the operators P_i such that $P_r^i(\omega_0) = P_r^i(\omega) = r_i$, thus

$$P_i |\omega_0, r\rangle = r_i |\omega_0, r\rangle, \quad P_i |\omega, r\rangle = r_i |\omega, r\rangle. \quad (14)$$

Therefore $\{|\omega_0, r\rangle, |\omega, r\rangle\}$ is the observers pointer basis where there is a perfect decoherence in the corresponding state co-basis. Moreover the generalized states $(\omega_0, rr'|$ and $(\omega, rr'|$ are constants of the motion, and therefore these exact pointer observables have a constant statistical entropy and will be "at the top of the list" of Zurek's "predictability sieve" [9].

Therefore:

i.- Decoherence in the energy is produced by the time evolution.

ii.- Decoherence in the other dynamical variables can be seen if we choose an adequate basis, namely the final pointer basis.

Our main result is eq. (12): *When $t \rightarrow \infty$ then $\rho(t) \rightarrow \rho_*$ and in this state the dynamical variables H, P_1, \dots, P_N are well defined. Therefore the eventual conjugated variables to these momentum variables (namely: configuration variables, if they exists) are completely undefined.*

In fact, calling by \mathbb{L}_i the generator of the displacements along the eventual configuration variable conjugated to P_i , we have $(\mathbb{L}_i \rho_* | O) = (\rho_* | \mathbb{L}_i^\dagger O) = (\rho_* | [P_i, O]) = 0$ for all $O \in \mathcal{O}$. Then ρ_* is homogeneous in these configuration variables.

From the preceding section we may have the feeling that the process of decoherence must be found in all the physical systems, and therefore, all of them eventually would become classical when $\hbar \rightarrow 0$. It is not so as explained in [7].

III. THE CLASSICAL EQUILIBRIUM LIMIT.

A. Expansion in sets of classical motions.

In this section we will use the Wigner integrals that introduce an isomorphism between quantum observables O and states ρ and their classical analogues $O^W(q, p)$ and $\rho^W(q, p)$ [8]:

$$\begin{aligned} O^W(q, p) &= \int d\lambda \langle q - \frac{\lambda}{2} | O | q + \frac{\lambda}{2} \rangle \exp(\frac{i\lambda p}{\hbar}) \\ \rho^W(q, p) &= \frac{1}{\pi\hbar} \int d\lambda (\rho || q + \lambda \rangle \langle q - \lambda |) \exp(\frac{2i\lambda p}{\hbar}). \end{aligned} \quad (15)$$

It is possible to prove that $\int dq dp \rho^W(q, p) = (\rho | I) = 1$, but ρ^W is not in general non negative. It is also possible to deduce that

$$(\rho^W | O^W) = \int dq dp \rho^W(q, p) O^W(q, p) = (\rho | O), \quad (16)$$

and therefore to the mean value in the classical Liouville space it corresponds the mean value in the quantum Liouville space.

Moreover, calling by L the classical Liouville operator, and by \mathbb{L} the quantum Liouville-Von Neumann operator, we have

$$L [\rho^W(q, p)] = [\mathbb{L}\rho]^W(q, p) + O(\hbar), \quad (17)$$

where $L \rho^W(q, p) = i \{H^W(q, p), \rho^W(q, p)\}_{PB}$ and

$$(\mathbb{L}\rho|O) = (\rho|[H, O]). \quad (18)$$

Finally, if $O = O_1 O_2$, where O_1 and O_2 are two quantum observables, we have

$$O^W(q, p) = O_1^W(q, p) O_2^W(q, p) + O(\hbar). \quad (19)$$

We will prove that the distribution function $\rho_*^W(q, p)$, that corresponds to the state functional ρ_* via the Wigner integral is a non negative function of the classical constants of the motion, in our case $H^W(q, p)$, $P_1^W(q, p), \dots, P_N^W(q, p)$, obtained from the corresponding quantum operators H, P_1, \dots, P_N .

From eq. (12) we have:

$$\rho_* = W \lim_{t \rightarrow \infty} \rho(t) = \sum_r \rho_r(\omega_0)(\omega_0, rr| + \sum_r \int_0^\infty d\omega \rho_r(\omega)(\omega, rr|, \quad (20)$$

so we must compute:

$$\rho_{\omega r}^W(q, p) \doteq \pi^{-1} \int (\omega, rr| |q + \lambda\rangle \langle q - \lambda|) e^{2ip\lambda} d\lambda \quad (21)$$

We know from [1] section II. C, (or we can prove directly from eqs.(12-14)) that

$$\begin{aligned} (\omega_0, rr| H^n) &= \omega_0^n, & (\omega, rr| H^n) &= \omega^n, \\ (\omega_0, rr| P_i^n) &= r_i^n, & (\omega, rr| P_i^n) &= r_i^n, \quad i = 1, \dots, N \end{aligned} \quad (22)$$

for $n = 0, 1, 2, \dots$ Using the relation (19) between quantum and classical products of observables and relation (16) between quantum and classical mean values, in the limit $\hbar \rightarrow 0$ (we will consider that we always take this limit when we refer to classical equations below) we deduce that the characteristic property of the distribution $\rho_{\omega r}^W(q, p)$, that corresponds to the state functional $(\omega, rr|$, is:

$$\int \rho_{\omega r}^W(q, p) [H^W(q, p)]^n dq dp = \omega^n, \quad \int \rho_{\omega r}^W(q, p) [P_i^W(q, p)]^n dq dp = r_i^n, \quad (23)$$

for any natural number n . Thus $\rho_{\omega r}^W(q, p)$ must be the functional

$$\rho_{\omega r}^W(q, p) = \delta(H^W(q, p) - \omega) \delta(P_1^W(q, p) - r_1) \dots \delta(P_N^W(q, p) - r_N). \quad (24)$$

For the distribution $\rho_{\omega_0 r}^W(q, p)$ corresponding to the state functional $(\omega_0, rr|$, we obtain

$$\rho_{\omega_0 r}^W(q, p) = \delta(H^W(q, p) - \omega_0) \delta(P_1^W(q, p) - r_1) \dots \delta(P_N^W(q, p) - r_N). \quad (25)$$

Therefore, going back to eq. (20) and since the Wigner relation is linear, we have:

$$\rho_*^W(q, p) = \sum_r \rho_r(\omega_0) \rho_{\omega_0 r}^W(q, p) + \sum_r \int_0^\infty d\omega \rho_r(\omega) \rho_{\omega r}^W(q, p). \quad (26)$$

Also we obtain $\rho_*^W(q, p) \geq 0$, because $\rho_r(\omega_0)$ and $\rho_r(\omega)$ are non negative.

Therefore, the classical state $\rho_*^W(q, p)$ is a linear combination of the generalized classical states $\rho_{x r}^W(q, p)$ (where x is either ω_0 or ω), having well defined values x, r_1, \dots, r_N of the classical observables $H^W(q, p), P_1^W(q, p), \dots, P_N^W(q, p)$ and the corresponding classical canonically conjugated variables completely undefined since $\rho_{x r}^W(q, p)$ is not a function of these variables. *So we reach, in the classical case, to the same conclusion than in the quantum case (see end of subsection 2. 2).* But now all the classical canonically conjugated variables a_0, a_1, \dots, a_N do exist since they can be found solving the corresponding Poisson brackets differential equations. We can also expand the densities given in eqs. (24-26) in terms of classical motions as shown in [7].

- i.- We have shown that the quantum state functional $\rho(t)$ evolves to a diagonal state ρ_* .
- ii.- This quantum state ρ_* has $\rho_*^W(q, p)$ as its corresponding classical density.
- iii.- This classical density can be decomposed in sets of classical motions where H^W, P_1^W, \dots, P_N^W remain constant. These motions have origins $a_0(0), a_1(0), \dots, a_N(0)$ distributed in an homogeneous way.
- iv.- From eqs. (24-26) we obtained that

$$\rho_*^W(q, p) = f(H^W(q, p), P_1^W(q, p), \dots, P_N^W(q, p)) \geq 0.$$

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